Advanced ODE-Lecture 3
Extensibility of Solution

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Outline

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- Extensibility of Solution
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Motivation

- Weak point of Peano Theorem and Picard Theorem: They are both local results and tell nothing about the information on length of existence for solution.

- Global result is good for applications. Solutions of IVP might not exist for all $t \in R$ even though the differential equation is defined for $t \in R$. This raises a question about maximal interval on which a solution can be defined. Extensibility result gives how it will be from the local to the global.

- Lipschitz condition is a sufficient condition for uniqueness of solution and how to verify is a technical concern.

Motivated Example:

$$\text{Riccati Equation: } \begin{cases} x' = t^2 + x^2 \\ x(0) = 0 \end{cases}.$$

Applying Peano (Picard) Theorem, we find

- $Q_1 = \{(t,x) : |t| \le 1, |x| \le 1\}$, $M = \max_{(t,x)\in Q_1} |t^2 + x^2| = 2 \Rightarrow h_1 = \min\{a, \frac{b}{M}\} = \frac{1}{2}$;
- $Q_2 = \{(t,x) : |t| \le 2, |x| \le 2\}$, $M = \max_{(t,x)\in Q_2} |t^2 + x^2| = 8 \Rightarrow h_2 = \min\{a, \frac{b}{M}\} = \frac{1}{4}$.

Some phenomenon arises: $Q_1 \subset Q_2$, but $h_1 > h_2$!

Observation:

- This example motivates us that the solution, which is ensured by Peano (Picard) Theorem, is extendable form $[-h_2, h_2]$ to $[-h_1, h_1]$;
- Peano (Picard) Theorem tells nothing about information on the length of existence of interval. We have to develop a new result to characterize extensibility property –**Extensibility Theorem**.

Extensibility of Solution

1) Some Notions

Definition 3.1 $f: G \to R^n$, where G is an open set of $R \times R^n$, is said to satisfy a **local Lipschitz condition** if for any $(t_0, x_0) \in G$, there exists a neighborhood $(t_0, x_0) \in U \subset G$ such that f satisfies a Lipschitz condition on U.

Remark 3.1 Pay attention on the difference between local Lipschitz condition and Lipschitz condition.

Definition 3.2 Let $x(t)$ be a solution of the IVP on (α, β). If there exists the other solution $\tilde{x}(t)$ of the IVP on $(\tilde{\alpha}, \tilde{\beta})$ such that

- $(\tilde{\alpha}, \tilde{\beta}) \supset (\alpha, \beta)$, but $(\tilde{\alpha}, \tilde{\beta}) \neq (\alpha, \beta)$;
- $\tilde{x}(t) \equiv x(t)$ for $t \in (\alpha, \beta)$,

we say that $x(t)\,(t \in (\alpha, \beta))$ is **extendable**, and $\tilde{x}(t)$ is said to be **extension** of $x(t)$ on $(\tilde{\alpha}, \tilde{\beta})$. We say that a solution $x(t)$ is **non-extendable** if no such extension exists. That is, (α, β) is a **maximal interval of existence** of $x(t)$. Denoted by $I_{\max} = (\omega_-, \omega_+)$.

2) Extensibility Process

Consider the IVP, where $f: G \to R^n$ is continuous and local Lipschitz. For the case where $t > t_0$ only, $t < t_0$ is similar.

- $\forall (t_0, x_0) \in G \Rightarrow$ The solution $x(t)$ exists on $I_0 := [t_0, t_0 + h_0]$ with $h_0 > 0$ by Peano theorem, so $x(t_1)$ with $t_1 = t_0 + h_0$ exists and $(t_1, x(t_1)) \in G$;
- If $(t_1, x(t_1)) \in G$ is an interior point of G, then we apply Peano theorem at this point once more and have a new interval $I_1 := [t_1, t_1 + h_1]$ with $h_1 > 0$, on which $x(t)$ exists. Therefore $x(t_2)$ with $t_2 = t_1 + h_1$ exists and $(t_2, x(t_2)) \in G$;

- If $(t_2, x(t_2))$ is an interior point of G, then we repeat the step 2 to get an interval $I_2 := [t_2,\ t_2 + h_2]$ with $h_2 > 0, \cdots$; to get $I_j := [t_j, t_j + h_j]$ with $h_j > 0$ on which $x(t)$ exists. Then $x(t)$ is now extended to $\overset{j}{\underset{k=1}{\cup}} I_k$;
- If G is open and bounded, I_j is smaller and smaller because $x(t) \to \partial G$, ∂G is a boundary of G ;

If G is closed and bounded (compact), the extension will be terminated for some step $j = k$ because $(t_k, x(t_k))$ is on ∂G, which cannot be applied by Peano theorem anymore.

Remark 3.2 The process shows that in all cases, $I_{\max}$ can be found. If G is open, which is usually assumed, then $I_{\max}$ must be open;

Remark 3.3 For f with different $(t_0, x_0) \in G$, $I_{\max}$ might be different! We hope to know what conditions assure the same $I_{\max}$ for all $(t_0, x_0) \in G$. This is a real concern in ODE, which is referred as a **global existence**!!

Remark 3.4 In some case, $x(t)$ will **blow up** at finite time (**finite escape**).

Example $\begin{cases} x' = x^2 \\ x(0) = 1 \end{cases}$ has a solution $x(t) = \dfrac{1}{1-t}$ with $\lim_{t \to 1^-} x(t) = \infty$, $I_{\max} = (-\infty, 1)$.

Remark 3.5 The process of extensibility is nothing special except for its asymptotic behavior of solution. This is a real concern of extensibility process.

3) Extensibility Theorem

Theorem 3.1 (Extensibility Theorem) Suppose that G is open in $R \times R^n$, $f : G \to R^n$ is continuous (local Lipschitz). Then every solution of IVP has extensibility up to the boundary of G. More precisely, if $x : I_{\max} = (\omega_-, \omega_+) \to R^n$ is a solution passing through $(t_0, x_0) \in G$, then for any compact set $K \subset G$ there exist t_1 and t_2 with $t_1 < t_0 < t_2$ such that $(t_1, x(t_1)) \neq K$, $(t_2, x(t_2)) \neq K$.

Remark 3.6 This theorem states that any solution starting at point in G can be extended continuously to ∂G, which can also be formulized as follows.

$$\lim_{t\to\omega_\pm}\{d(P(t),\partial G)^{-1}+\|P(t)\|\}=\infty, \tag{F1}$$

where $P(t)=(t,x(t))$; d is a distance between $p(t)$ and ∂G; $\|p(t)\|=(t^2+x^2(t))^{\frac{1}{2}}$.

If $G=R\times R^n$, then ∂G is an empty set. i.e. $d(P(t),\partial G)^{-1}=0$, (F1) becomes

$$\overline{\lim_{t\to\omega_\pm}}\|P(t)\|=\infty.$$

It means that either $I_{\max}=(-\infty,\infty)$ **(global existence)** or if $I_{\max}=(\omega_-,\omega_+)$, where $\omega_+<\infty$ and $\omega_->-\infty$, then $\overline{\lim_{t\to\omega_\pm}}\|x(t)\|=\infty$ **(finite escape)**.

Proof of Extensibility Theorem. We only prove the case of $[t_0, \omega_+)$.

If $\omega_+ = \infty$, then $\exists$ $t_2 > t_0$ s.t. $(t_2, x(t_2)) \in K$ because K is bounded in $G = R \times R^n$. If $\omega_+ < \infty$. Show by contradiction. If $\exists$ a compact $K \subset G$ s.t. $(t, x(t)) \in K$ for all $t \in [t_0, \omega_+)$. Since f is bounded (say M) on K, then we have

$$\| x(t) - x(\tilde{t}) \| \leq \left| \int_{\tilde{t}}^{t} \| f(s, x(s)) \| \, ds \right| \leq M \, | t - \tilde{t} | .$$

So $x(t)$ is uniformly continuous on $[t_0, \omega_+)$. Then, $x(\omega_+) = \lim_{t \to \omega_+} x(t)$ exists and is finite. Moreover, $(\omega_+, x(\omega_+)) \in K$ because K is closed. Then,

$$(\omega_+, x(\omega_+)) \in K \subset G$$

is an interior point of G, which shows that it is extendable at ω_+ by Peano (Picard) theorem. This contradicts the maximality of $I_{\max}$. □

Corollary 3.2 (Extensibility Theorem II) If $f(t,x) \in C(G)$, where $G \subset R^{n+1}$ is a bounded domain, then, for $x = x(t)$, $t \in I_{\max} = (\omega_-, \omega_+)$, we have

$$\overline{\lim_{t \to \omega_+^-(\omega_-^+)}} d(P(t), \partial G) = 0 .$$

Corollary 3.3 (Extensibility Theorem III) If $f(t,x) \in C(R^{n+1})$, then, for $x = x(t)$, $t \in I_{\max} = (\omega_-, \omega_+)$ it is alternative as follows.

- $\omega_- = -\infty$ $(\omega_+ = \infty)$; or
- $\omega_- > -\infty$ $(\omega_+ < \infty)$, then $\lim_{t \to \omega_-^+(\omega_+^-)} \| x(t) \| = \infty$.

Applications of Extensibility

Example 3.1 If $x' = f(t,x)$, where $f \in C$ and $\| f(t,x) \| \leq M$ for all $(t,x) \in R \times R^n$, show that for any (t_0, x_0), the solution $x(t)$ has $I_{\max} = (-\infty, \infty)$.

Proof. For any (t_0, x_0), we have $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$, and then

$$\| x(t) \| \leq \| x_0 \| + \left| \int_{t_0}^{t} \| f(s, x(s)) \| \, ds \right| \leq \| x_0 \| + M \, | t - t_0 | .$$

Show by contradiction. If $t_0 \leq t < \omega_+$ with $\omega_+ < \infty$, then

$$\| x(t) \| \leq \| x_0 \| + M(\omega_+ - t_0) < \infty \quad \Rightarrow \quad \overline{\lim_{t \to \omega_+}} \, \| x(t) \| < \infty .$$

This contradicts Extensibility theorem. It must have $\omega_+ = \infty$. It is similar to show the case of $\omega\_ < t \leq t_0$ with $\omega\_ > -\infty$. □

Example 3.2 All solutions of the Riccati equation $x' = t^2 + x^2$ have a finite escape.

Proof. Only show $[t_0, \omega_+)$ with $\omega_+ < \infty$. It is similar to show $\omega_- < t \le t_0$ with $\omega_- > -\infty$. If $\omega_+ \le 0$, then, $\omega_+ < \infty$. If $\omega_+ > 0$, then there exists $t_1 > 0$ such that $[t_1, \omega_+) \subseteq [t_0, \omega_+)$. Then we have

$$x'(t) \ge t_1^2 + x^2(t), \quad t \in [t_1, \omega_+) \quad \Leftrightarrow \quad \frac{dx(t)}{t_1^2 + x^2(t)} \ge dt, \quad t \in [t_1, \omega_+).$$

Integration on both sides, we obtain

$$\frac{1}{t_1}[\arctan\frac{x(t)}{t_1} - \arctan\frac{x(t_1)}{t_1}] \ge t - t_1 \ge 0, \quad t \in [t_1, \omega_+).$$

From the above it yields $0 \le t - t_1 \le \frac{\pi}{t_1}$, $t \in [t_1, \omega_+)$. That is, $0 < \omega_+ \le t_1 + \frac{\pi}{t_1} < \infty$. □

Comments on Lipschitz Condition

Definition 3.3 $f: G \to R^n$, where G is open in R^{n+1}, is said to satisfy a **local Lipschitz condition** if for any $(t_0, x_0) \in G$, there exists a neighborhood $(t_0, x_0) \in U \subset G$ such that f satisfies a Lipschitz condition on U. If $U = R^{n+1}$, the corresponding Lipschitz condition is said to be **global Lipschitz**.

Remark 3.7 It is not easy in general to verify the Lipschitz condition by definition. However, if $\frac{\partial f}{\partial x}(t,x)$ is continuous on Q, then, we can take

$$L \geq \max_{(t,x)\in Q} \left\| \frac{\partial f}{\partial x}(t,x) \right\|,$$

where $\frac{\partial f}{\partial x} = \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,n}$ is the Jacobian matrix of f. Therefore,

$$\frac{\partial f}{\partial x} \text{ is continuous on } Q \Rightarrow f \text{ is Lipschitz on } Q.$$

However, the opposite is not true! e.g. $f(t,x) = |x|$ at $x = 0$.

Remark 3.8 How to test that f is not Lipschitz?

We restrict us in R. It is similar in R^n. If $\frac{\partial f}{\partial x}(t,x)$ exists except for $x = x_0$, and $\lim_{x \to x_0} \frac{\partial f}{\partial x}(t,x) = \infty$, then $f(t,x)$ doesn't satisfy Lipschitz condition on any Q (or U) containing $x = x_0$.

In fact, since $\lim_{x \to x_0} \frac{\partial f}{\partial x}(t,x) = \infty$, one has $\lim_{x \to x_0} \frac{f(t,x) - f(t,x_0)}{x - x_0} = \infty$. Then, for any given $K > 0$, there exists $\delta > 0$, such that

$$| f(t,x) - f(t,x_0) | > K | x - x_0 |,$$

whenever $| x - x_0 | < \delta$. Therefore, we cannot find Lipschitz constant L in any domain containing $x = x_0$.

Remark 3.9 A Lipschitz condition is a sufficient condition for uniqueness! It doesn't say anything for uniqueness if it is not Lipschitz. See an example as follows.

$$x' = f(t,x) = \begin{cases} x\ln|x| & x \neq 0 \\ 0 & x = 0 \end{cases},$$

where $f(t,x)$ is continuous on R^2 and is not Lipschitz on any domain containing $x = 0$ (Homework). However, its explicit solution is solved as

$$x = \pm\exp\{ce^t\} \text{ and } x = 0.$$

For any initial value $(t_0, x_0) \in R^2$, there exists a unique curve passing through (t_0, x_0).

Summary

- Extensibility Theorem is a bridge connecting local and global.
- How to apply Extensibility Theorem is a main concern in the sequel.
- Local Lipschitz is a mild condition and most physical models have such a property. But global Lipschitz is a restrict one and even linear time-varying systems may not satisfy it.
- How to verify the Lipschitz is a skillful work.

Homework

1) Solve the example in Remark 3.9.

2) Prove Extensibility Theorem II and III.

3) If $x' = f(t,x)$, where $f \in C$ and $\|f(t,x)\| \le a\|x\| + b$ for all $(t,x) \in R^{n+1}$, show that for any $(t_0, x_0) \in R^{n+1}$, the solution $x(t)$ passing through (t_0, x_0) has $I_{\max} = (-\infty, \infty)$.

4) Review today's class.